Thus these points are related to this branch of the locus in the same way as the origin is related to the conchoid $(x^2 + y^2)(x - a)^2 = b^2x^2$.

The coördinates (0, 0) always satisfy this equation, and when $y = 0$, we get $x = 0, 0, a \pm b$, but the curve does not cut the x-axis four times unless $b > a$. That is, the origin is a double point, or a conjugate point according as $b > a$, or $b < a$. The same thing is true in regard to the limaçon

$$(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

NOTE ON THE INCENTERS OF A QUADRILATERAL.

By F. V. Morley, Johns Hopkins University.

The rectangular configuration of the incenters of an inscribed quadrilateral was known to Professor Neuberg of Liège before 1906.[1] It was rediscovered, as an isolated fact, by the writer in 1914, and several proofs were supplied by his father, Professor Morley. When the Monthly enlarged its scope in 1916 the proposition was sent in as a problem, appearing in March, 1917; the obvious extension and the writer's solution were published soon after.[2] In June, 1918, Professor Altshiller-Court published an article in this Monthly, rediscovering the configuration.[3]

The proposition as an isolated fact with reference to the inscribed quadrilateral was therefore worth discovery; but it will now be shown that the proposition appears in a well-known theory.

I.

There are several beautiful chains of theorems concerned with the elementary geometry of n-lines in a plane. One of these is Clifford's chain, and others of similar type are well known to students of metric or reflexive geometry. With slight modifications, they apply to directed as well as to undirected lines.[4]

One of these chains is concerned with the incenters of n-lines. In the present note it will be simpler to consider the lines as directed. For example, a triangle composed of three undirected lines has four incenters, using the term in its general meaning; but if the three lines are thought of as directed, there is only one circle tangent to all three in the proper sense, and hence only one incenter.

[1] The reference here seems to be to Neuberg's article in *Mathesis*, 1906, pp. 14–17. But Neuberg published the result as a problem many years earlier (*Nouvelle correspondance mathématique*, Tome 1, 1875, p. 96), accompanied by the statement that it was extracted from *Archiv der Mathematik und Physik*, 1842, p. 328. Catalan's solution of the problem appears in *Nouvelle correspondance*, tome 1, pp. 198–200.—Editor.

[2] November, 1917, volume 24, pp. 429–430.

[3] Volume 25, 1918, pp. 241–246; for comment, see volume 26, 1919, pp. 65–66. Still more recently, the subject has been discussed in the comprehensive article by J. W. Clawson, in the *Annals of Math.*, vol. 20, p. 254 (1919). [Mr. Morley's paper was in the hands of the Editor some time before Professor Clawson's paper was published.—Editor.]

[4] Cf. F. Morley, *Transactions of the American Mathematical Society*, Vol. 1, pp. 97–115; and F. H. Loud, *Transactions of the American Mathematical Society*, Vol. 1, pp. 323–338.

Four directed lines may be taken three at a time in four ways; each triangle will have its incenter, and according to the theorem of Steiner which is the basis of the chain, these four incenters lie on a circle.[1] This circle has been called the center-circle[2] of the quadrilateral; its center is a unique point of the quadrilateral, and may be called by analogy its incenter. The notation commonly used is C_4 for the center-circle of four directed lines, and c_4 for the incenter.

The statement of the chain proceeds as follows.[3]

1. *Five directed lines have five points c_4, the incenters of the lines taken four at a time; these five points are on a circle, C_5, whose center, c_5, is the incenter of the 5-line. Six directed lines have six points c_5, on a circle C_6; and in general, n directed lines have n points of the type c_{n-1}, which are on a circle C_n, the center of which is the incenter, c_n, of the n-line.*

2. *Five directed lines have five circles C_4; these circles meet in a point, N_5, called the node of the 5-line; and in general, n directed lines have n circles C_{n-1} on a node N_n.*

These general theorems apply to the circles C_{n-1} and the centers c_{n-1} of n lines. The configuration has associated with it a variety of other circles, C_{n-2}, C_{n-3}, . . . , with corresponding centers, on which there are definite conditions; but the number and complexity of these in general makes their investigation of doubtful value. Special cases occur, however, in which the subcircles and centers simplify, and one of these has recently aroused some interest.

II.

In dealing with four directed lines, the reversal of any one direction will produce an entirely different arrangement of the circles and centers. The same directed quadrilateral will have all the features of five, six, seven, or eight lines, according to the number of its sides which are counted both ways and as the general configuration will be considerably simplified, new properties of the quadrilateral may be emphasized.

Eight directed lines will have 56 points of the type c_3, four at a time on 70 circles C_4. There will be 70 centers c_4, five at a time on 56 circles C_5. The 56 centers c_5 are six at a time on 28 circles C_6; the 28 centers c_6 are seven at a time on 8 circles, C_7 whose centers c_7 form the center-circle of the 8-line. So much the general theorem prescribes; and when the quadrilateral, by reversing its sides, is considered as eight lines, these facts will hold. The centers will double up, however, and there will appear only 28 c_3, 38 c_4, 28 c_5, 16 c_6, 4 c_7; moreover, not all of

[1] Stated without proof in *Annales de mathématiques* (Gergonne), Tome 18, 1828, p. 302. A proof was furnished by J. Mention in *Nouvelles annales de mathématiques*, vol. 21, 1862, p. 16f. The theorem did not, however, originate with Steiner but with L. Puissant; see *Correspondance sur l'Ecole Polytechnique*, Tome 1, 1806, p. 193.—EDITOR.

[2] The term "*center-circle*" was used by F. Morley, *l.c.*, p. 99, for the circle on the circumcenters of four lines taken three at a time. But later, *Trans. Am. Math. Soc.*, vol. 8 (1907), p. 20, he alters the name for the circle on the circumcenters to "*centric-circle*." Now four directed lines have two distinct circles, one concerned with circumcenters (the centric-circle) and one on the four incenters. This last is the one with which we are concerned, and it is now called the "*center-circle*." Cf. Hodgson, *Trans. Am. Math. Soc.*, (1912), p. 203.

[3] Loud (*l.c.*), p. 325.

these are distinct, for the centers c_7 coalesce and unite with the incenter of the configuration. The points c_6 will then lie on a circle, and the points c_5 will be affected somewhat.

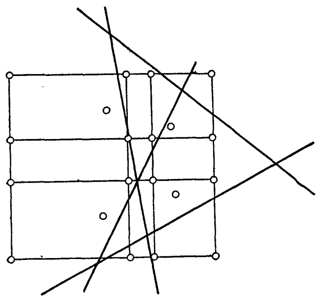

Fig. 1.

It is worth investigation to see how these comparatively few points c_5 behave in the case where the quadrilateral is considered as a special case of eight directed lines; and it is interesting to note in Figure 1 their resemblance to the configuration discussed above; namely, the rectangular net formed by the incenters of an inscribed quadrilateral.

The figure of the circles, C_5, is also of interest in this case, (Figure 2). As their centers are constrained to the rectangular net it is to be expected that they are in families of four; in addition they all pass through a point.

III.

The configuration suggested by the figures is readily proved. It is convenient to consider first a quadrilateral with two sides counted both ways; by repeating this figure the whole configuration may be obtained, but all that is essential is contained in Figure 3.

Here there are six directed lines. By the general theorem there is a center-circle for all six, expressed in the manner of Loud (*l.c.*, p. 325), as

$$x = a_0 - a_1 t$$

where the constants are complex and t an orthogonal number. The six circles C_5 are included in the double infinity of circles

$$x = a_0 - a_1(t + t') + a_2 tt'$$

which are known to pass through a point. Figure 2 is thus confirmed; all the circles C_5 pass through a point, which turns out to be the Clifford point of the quadrilateral.

On examination the six points c_5 in Figure 3 are seen to fall into two sets; four on a rectangle, and two separate. Consider now a triangle with two sides counted both ways; by drawing the centers c_4 it is found that the incenter c_5 coincides with the circumcenter of the triangle. Here there are two such triangles, formed by omitting the single lines 3 and 4 in turn. It follows that the two separate points noticed above are the circumcenters of those triangles, and in passing to the case of Figure 1, the four points c_5 which do not lie on the rectangular net are found to be the circumcenters of the four triangles of the original quadrilateral. They each count for 3 of the c_5, making up with the other 16 the total of 28. It is well known that the circumcenters of a quadrilateral lie on a circle;[1] in this particular case the node or Clifford point is also on this circle.

[1] Steiner, *Gesammelte Werke*, Band 1, p. 323.

That the other four centers in Figure 3 are rectangular, may be proved from the fact that all the circles of which they are centers pass through the node. Through the node there are a pair of natural rectangular axes,[1] indicated in the figure, to which the rectangle is parallel. These axes govern the grouping of the circles in Figure 2.

In order to identify the rectangular net of Figure 1 with the net of incenters of an inscribed quadrilateral, the 16 points must be incenters of the four triangles whose centers have been noted as on a circle. To prove this, draw any of the circles C_6, and note that the four points c_5 corresponding to each of the triangles of the inscribed quadrilateral are mutually orthocentric.[2]

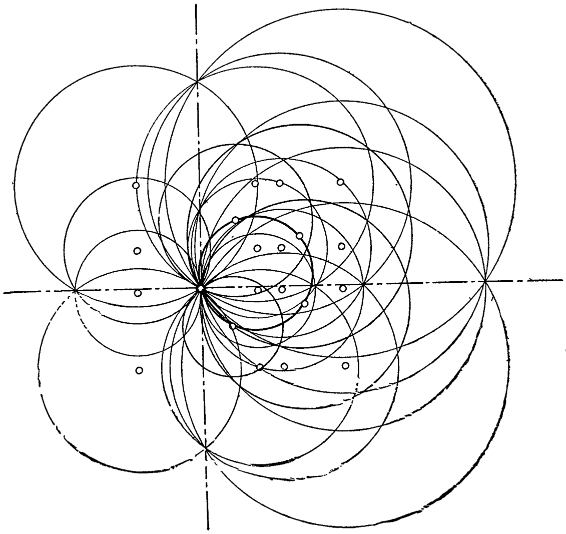

Fig. 2.

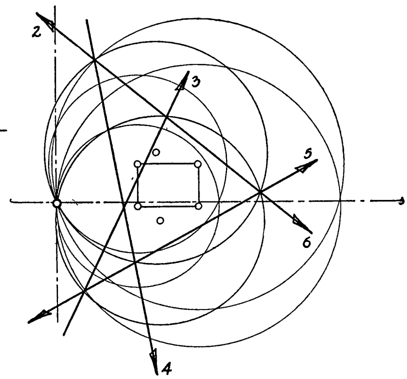

Fig. 3.

The rather pretty configuration resulting from the consideration of the quadrilateral as a special case in the theory of directed lines suggests the investigation of other cases. The centers c_4 of three lines counted both ways, for example, reduce to the nine point arrangement on a circle of Feuerbach. Similar configurations will hold for five or more lines, but the complication will increase. In the general treatment the analytical method of Professor Morley's memoir (*l.c.*), (which is followed in the paper by Dr. Loud (*l.c.*)), involves much less fatigue in the transference of thought than does the geometrical argument as here given.

---

[1] These are called the *incentric lines* by Clawson, l.c., p. 246.

[2] For a study of the properties of an orthocentric group of points see the article by Professor Altshiller-Court in the last number of this MONTHLY, *On the Orthocentric Quadrilateral*, pp. 199–202. —EDITOR.